FOUNDATION OF MATHEMATICS

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ABSTRACT. This is the lecture note of "Foundation of mathematics" [Math 2001], Spring 2026, HKUST.

1. Overview

The course starts with informally Sets theory. The foundation of Sets theory is omitted. The common Sets we use in the lecture are the set of natural numbers $\mathbb{N}$, the set of rational numbers $\mathbb{Q}$, the set of real number $\mathbb{R}$, and the set of complex number $\mathbb{C}$. The foundation of mathematics is indeed a huge subject. It is hard to address in several lectures. I basically divide the lecture to several topics. I can only give a brief introduction to different topics due to the limit of time. I summarize some topics that shape the modern Mathematics,

(1) Arithmetic of numbers.
(2) Analysis over $\mathbb{R}$, and $\mathbb{C}$.
(3) Topology, metric space.
(4) Algebra, linear algebra.

The Mathematics is so complicated that different subjects would interact, and are beneficial from each other.

2. Lecture 1

The goal of the first lecture is to understand the mathematical statement and proofs. Let's start with a simple example.

Theorem 2.1. *Let $a, b, c \in \mathbb{R}$, and let $a \neq 0$. Let $x \in \mathbb{R}$ be a root of the following equation $ax^2 + bx + c$. Then,*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let's analyze the theorem. First, we have some conditions for the statement:

(1) $a, b, c, x \in \mathbb{R}$, and $a \neq 0$.
(2) x is a root of the equation $ax^2 + bx + cx = 0$.

Note that we omit the definition of the real numbers. Next, the conclusion of the statement is the algebraic expression of the real number x,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Date: Spring, 2026.

Proof. First, we have an equation $ax^2 + bx + c = 0$. Since $a \neq 0$, we have $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Next, we have the equivalent expressions,

$$(x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0.$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}.$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

□

Question 2.2. Is the statement still true if change $\mathbb{R}$ to $\mathbb{Q}$?

Theorem 2.3. *The number* $\sqrt{2}$ *is not a rational number.*

Proof. Suppose $\sqrt{2}$ is a rational number. There are integers p, q, $gcd(p, q) = 1$ such that $\sqrt{2} = \frac{p}{q}$. Then,

$$2q^2 = p^2.$$

This implies $p = 2k$ for some integer k. Then,

$$q^2 = 2k^2$$

This implies $q = 2t$ for some integer t. It contradicts $gcd(p, q) = 1$. □

From the example, we see the mathematical statement will be false under different conditions.

2.1. **Set theory.** We introduce the informal theory of Sets. A set A is a collection of objects. The basic examples are the set of integer $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, the set of real number $\mathbb{R}$. The following is a list of Axiom(Definition) of Sets.

(1) Axiom of existence of empty set $\varnothing$. A set with no elements. Subset of any set.
(2) Axiom of equality. Let A and B be two sets. We say $A = B$ if for any $x \in A$, $x \in B$, and for any $x \in B$, $x \in A$.
(3) Axiom of union. Given sets A, B, there exists a set $A \cup B$ with the following properties,

$$x \in A \cup B \iff x \in A \text{ or } x \in B$$

(4) Axiom of subset. Consider the set A, and a condition $\mathcal{P}$ of element of A. There is a subset $B \subset A$,

$$B = \{x \in A | P(x)\}.$$

(5) Axiom of Power set. Let A be a set. There exists a set $\mathcal{P}(A)$ whose elements are all subsets of A.
(6) Definition of intersection.

$$A \cap B = \{x \in A |\ x \in B\}$$

(7) Definition of difference.

$$A \setminus B = \{x \in A | x \notin B\}$$

(8) Definition of complement. Let U be a universal set with subset A. The complement of A in U is

$$A^c = \{x \in U | x \notin A\}.$$

We usually omit the universal set U.

Let $A = \{1, 2, 3\} \subset \mathbb{Z}$, and let $B = \{1, 2, 2, 4, 5\} \subset \mathbb{Z}$.

- $B = \{1, 2, 4, 5\}$.
- $A \cup B = \{1, 2, 3, 4, 5\}$.
- $A \cap B = \{1, 2\}$.
- $A \setminus B = \{3\}$.
- $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- Let $U = \mathbb{Z}$. Then $A^c = \{x \in \mathbb{Z} | x \neq 1, 2, 3\}$

Lemma 2.4. *Let A and B be sets. We have,*

- $A \cap B = B \cap A$.
- $A \cup B = B \cup A$.
- $A \subset A \cup B$.
- $A \cap B \subset A$.
- $(A \cap B) \cap C = A \cap (B \cap C)$.
- $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof. Use the axiom of equality to check. □

Here are some useful properties of the operation of sets. Let A, B and C be sets in a universal set U.

Theorem 2.5. *Given sets A, B, and C, we have*

- $A \setminus (B \cup C) \subset (A \cup B) \setminus C$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $(A \cup B)^c = A^c \cap B^c$.
- $(A \cap B)^c = A^c \cup B^c$.

Proof. Exercise. □

Definition 2.6. (Cartesian product) Let A, B be sets. We have a set $A \times B$, referred to Cartesian product of A and B.

$$A \times B = \{(a, b) | a \in A; b \in B\}.$$

Remark 2.7. The pair $(-, -)$ is an order pair. Namely, let $a, c \in A$, $b, d \in B$. Then, $(a, b) = (c, d)$ if and only if $a = c, b = d$.

Example 2.8. Let $A = \{1, 2\}$, $B = \{2, 3\}$. Then,

$$A \times B = \{1, 2\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}.$$

Definition 2.9. A relation R of A and B is a non empty subset of $A \times B$. In the product $A \times A$,

(1) $R \subset A \times A$ is reflexive if $(a, a) \in R$.
(2) $R \subset A \times A$ is symmetric if $(a, b) \in R \implies (b, a) \in R$.

(3) $R \subset A \times A$ is transitive if for any element $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R$.
(4) R is an equivalence if R is reflexive, symmetric and transitive.

Example 2.10. In Example 2.8, the subset $\{(1,2)\}$ is a relation of A and B.

Example 2.11. Let $A = \{1,2,3\}$. The relation $R = \{(1,2),(2,1)\}$ is symmetric, but not transitive. The relation $R = \{(1,2),(2,3),(1,3)\}$ is transitive but not symmetric.

Definition 2.12. Let A and B be sets. Let R be a relation of A and B.

(1) If for any $a \in A$, there exists a unique $b \in B$ such that $(a,b) \in R$, then we say R define a function(map) from A to B. Usually we write $f : A \to B$, $a \mapsto f(a)$. The set $R = \{(a, f(a)) | a \in A; f(a) \in B\}$. The set A is the domain of function f.
(2) Let f and g be two functions from A to B. We say $f = g$ if for any $x \in A$, $f(x) = g(x)$.
(3) Let f be a function from A to B. We say f is injective if the following is true: for any element $x_1, x_2 \in A$, $f(x_1) = f(x_2) \implies x_1 = x_2$.
(4) Let f be a function from A to B. We say f is surjective if for any $b \in B$, there exists an element $a \in A$ such that $f(a) = b$.
(5) Let $C \subset A$, and let $f : A \to B$ be a function. The function $f : C \subset A \to B$ is the restriction of f to C. It is denoted as $f|_C$.
(6) The natural function $f : A \to A, a \mapsto a$ is the identity function.

Question 2.13. In Example 2.10, is the relation R defines a function from $\{1,2\}$ to $\{2,3\}$?

Question 2.14. In Example 2.8, we define a relation $R = \{(1,2),(1,3),(2,2)\}$. Does the relation defines a function from $\{1,2\}$ to $\{2,3\}$?

Example 2.15. Let $C \subset A$. We have a natural function

$$f : C \to A, c \mapsto c.$$

The function f is an injective function. It is surjective if and only if $C = A$.

Example 2.16. Fix some rational numbers $a, b \in \mathbb{Q}, a \neq 0$. Consider the linear function $f : \mathbb{Q} \to \mathbb{Q}, x \mapsto f(x) = ax + b$. The function f is injective and surjective.

Proof. Injectivity: for any x_1,x_2,

$$ax_1 + b = ax_2 + b \implies x_1 = x_2.$$

Surjectivity: for any $y \in \mathbb{Q}$, there exists $x = \frac{y-b}{a}$ such that $f(x) = y$. □

3. Lecture 2

First, I give a basic arithmetic property of the set of integers.

3.1. **Arithmetic of natural number.** Recall the set of natural numbers is

$$\mathbb{N} = \{0,1,2,3,\cdots\}.$$

We have commutative binary operations "+" and ".". Let $a, b, c \in \mathbb{N}$, we have,

$$a + b = b + a$$
$$a.b = b.a$$

Recall we have,

$$a.(b + c) = a.b + a.c$$

We have an order of elements in $\mathbb{N}$. Namely,

$$0 < 1 < 2 < \cdots$$

Let $p, q \in \mathbb{N}$. We write $p|q$ if there is a natural number k such that $q = pk$. For example, $2|6$. We say p is a divisor of q.

Definition 3.1. Let $p, q \in \mathbb{N}$.

(1) we say p and q are co-prime if there are no common divisor of p and q except 1.
(2) we say p is a prime number if p have no divisor except 1 and p.

Theorem 3.2. *Let p, q be natural numbers. Assume $q \geq p$. There exist unique k and r such that*

$$q = kp + r,\ 0 \leq r < p.$$

Proof. Consider the set $A = \{k | q - kp \geq 0\} \subset \mathbb{N}$. Then A is non-empty, and bounded above. Therefore, we have a maximal $k \in A$. Let $r = q - kp \geq 0$. If $r \geq p$, then $r - p = q - (k+1)p \geq 0$. Hence $k + 1 \in A$, contradicts that k is maximal. If there is a different pairs (k_1, r_1), (k_2, r_2) with

$$q = k_i p + r_i,\ i = 1, 2.$$

Then take a difference of two equalities, we have

$$|(k_1 - k_2)p| = |r_1 - r_2|.$$

But $|r_1 - r_2| \leq p - 1 < p \cdot |k_1 - k_2|$, a contradiction. □

Example 3.3. Let $p = 3$, $q = 7$. We have $7 = 2 \times 3 + 1$.

Corollary 3.4. *Let q and p be co-prime natural numbers. There exists integers s and t such that $p \cdot s + q \cdot t = 1$.*

Proof. Without loss of generality, we assume $q > p$. According to Theorem 3.2, there exists k_1, r_1, $0 < r_1 < p$,

$$q = k_1 \cdot p + r_1.$$

Since p and q are co-prime, p and r_1 are co-prime. Similarly, there exists k_2, r_2, $0 \leq r_2 < r_1$,

$$p = k_2 \cdot r_1 + r_2.$$

Continue the procedure, for any $n \geq 1$, there exists k_n, r_n, $0 < r_n < r_{n-1}$,

$$r_{n-2} = k_n \cdot r_{n-1} + r_n.$$

Here, we write $r_{-1} = q$ and $r_0 = p$. Consider the set $A = \{r_{-1}, r_0, \cdots, r_n \cdots, \} \subset \mathbb{N}$. Since $r_{-1} > r_0 > r_1 > \cdots \geq 1$, the set A has finitely many elements with minimal element 1. Therefore, there exists some integer m such that $r_{m-1} = k_{m+1} \cdot r_m + 1$. Then, Since r_i can be expressed as of the form $- \cdot p + - \cdot q$, 1 can be expressed as the form $- \cdot p + - \cdot q$. □

Example 3.5. Let q=11, p=3. Then, $11 = 3 \cdot 3 + 2$, $3 = 2 + 1$. Thus,

$$\begin{aligned} 1 &= 3 - 2 \\ &= 3 - (11 - 3 \cdot 3) \\ &= 4 \cdot 3 - 11 \end{aligned}$$

Remark 3.6. Such pair (s, t) is not unique. For example, $1 = 4 \times 3 - 11 = 2 \cdot 11 - 7 \cdot 3$.

Corollary 3.7. *Let q and p be natural numbers. Let m be the maximal divisor of p and q, which is denoted as $gcd(p, q)$. There exists s and t such that $s \cdot q + t \cdot p = m$.*

Definition 3.8. Let q and p be integers, and let m be a positive natural number. We write $q \equiv p(mod\ m)$ if there exists an integer k such that $q - p = km$.

Example 3.9. Take $m = 3$, $-1 \equiv 8 \equiv 11(mod\ m)$.

Lemma 3.10. *Let a, b, c, d be integers. Let m be a positive natural number. Assume $a \equiv b(mod\ m)$, $c \equiv d(mod\ m)$.*

(1) *$a \cdot c \equiv b \cdot d(mod\ m)$.*
(2) *For any integer n, $a \pm n \equiv b \pm n(mod\ m)$.*
(3) *For any positive integer t, $a^t \equiv b^t(mod\ m)$.*

Question 3.11. Let $q = 2^{2026}$, $p = 5$. By Theorem 3.2, there exists unique k and $0 \leq r < p$ such that $2^{2026} = k \cdot 5 + r$. What is the number r?

Proof. Exercise. □

Theorem 3.12. *(Chinese remainder theorem)*
Let $\{p_1, p_2, \cdots, p_k\}$ be a collection of positive natural numbers that are co-prime to each other. Pick a collection of integers $\{a_1, a_2, \cdots, a_k\}$ such that $0 \leq a_i < p_i - 1$. There exists a unique $0 \leq x < p_1 \cdot p_2 \cdots p_k$ such that $x \equiv a_i(mod\ p_i)$

Proof. Let $N_i = \prod_{j \neq i} p_j$. Since N_i is coprime to p_i, by Corollary 3.7, there are integer M_i and m_i,

$$M_i \cdot N_i + m_i \cdot p_i = 1.$$

Let $x = \sum_{i=1}^{k} a_i \cdot M_i \cdot N_i$. Then $x' \equiv a_i \cdot (1 - m_i \cdot p_i) \equiv a_i(mod\ p_i)$. Add some multiple of $p_1 \cdot p_2 \cdots p_k$, we have some $x \equiv a_i(mod\ p_i)$, and $0 \leq x < p_1 \cdot p_2 \cdots p_k$.

We show uniqueness. Suppose there are two number x_1 and x_2 satisfies the property. Then, $x_1 - x_2 \equiv 0(mod\ p_i)$. Therefore $x_1 - x_2$ is a multiple of $p_1 \cdot p_2 \cdots p_k$. However, $|x_1 - x_2| < p_1 \cdot p_2 \cdots p_k$, a contradiction. □

Example 3.13. Find the integer $0 \leq x < 30$,

$$x \equiv 1(mod\ 2)$$
$$x \equiv 2(mod\ 3)$$
$$x \equiv 1(mod\ 5).$$

Proof. Use the notation in the Theorem 3.12, we write $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $a_1 = 1$, $a_2 = 2$, and $a_3 = 1$. Then $p_1 \cdot p_2 \cdot p_3 = 30$. Simple calculation shows $M_1 = 1$, $M_2 = 1$, $M_3 = 1$. Let $x = \sum_{i=1}^{3} a_i M_i \cdot N_i = 15 + 2 \cdot 10 + 6 = 41$. Thus, the number we find is 11.

□

Example 3.14. Find the integer $0 \leq x < 105$,

$$x \equiv 1(mod\ 3)$$
$$x \equiv 2(mod\ 5)$$
$$x \equiv 3(mod\ 7).$$

Proof. Exercise. □

4. Lecture 3

In this lecture, we study the construction of rational numbers. We begin with orderd sets.

Definition 4.1. Let A be a set. An order in A is a relation $<$ with the following two properties,

(1) If $x, y \in A$, then one and only one of the following is true,

$$x < y;\ \ x = y;\ \ y < x.$$

(2) The relation $<$ is transitive.

Example 4.2. The relation $<$ is an order in the set of natural numbers $\mathbb{N}$.

Definition 4.3. Let A be a set with order $<$. Let $B \subset A$. We say B is bounded above if there exist an element $a \in A$ such that $b \leq a$ for any $b \in B$. a is an upper bound of A. The definition bounded below and lower bounded are defined similarly(replace $\leq$ to $\geq$).

Example 4.4. Let $B = \{1, 2, 3, 7, 9\} \subset \mathbb{N}$. Then 10 is an upper bound of B, 1 is a lower bound of B.

Definition 4.5. (least upper bound) Let A be an order set. Suppose a subset B is bounded above. If there exists an "minimal" upper bound a in the following way: for any upper bound b of B, $b \leq a$, then a is the least upper bound. We write

$$a = Sup\ B.$$

Similarly for the definition of greatest lower bound. We write the great lower bound as $inf\ B$

Lemma 4.6. *The least upper bound is unique.(greatest lower bound)*

Proof. Let a_1 and a_2 be the least upper bounds of B. Since a_1 is a least upper bound, and a_2 is an upper bound, $a_2 \leq a_1$. Since a_2 is a least upper bound, and a_1 is an upper bound, $a_2 \leq a_1$. Thus $a_1 = a_2$. □

Example 4.7. Let $B = \{1, 2, 3, 7, 9\} \subset \mathbb{N}$. Then 9 is the least upper bound of B, and 1 is the greatest lower bound of B.

Definition 4.8. *A* field F is a set with two operation "+" and "·", with the following properties,

(1) (A.1) The addition + is commutative: $x + y = y + x$.
(2) (A.2) The addition is associative: $x + (y + z) = (x + y) + z$.
(3) (A.3) There exist a unit 0 with $x + 0 = x$.
(4) (A.4) For any x, there is $(-x)$ with $x + (-x) = 0$.
(5) (M.1) The multiplication · is commutative: $x \cdot y = y \cdot x$.
(6) (M.2) The multiplication is associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
(7) (M.3) There is a unit $1 \neq 0$ with $(x \cdot 1 = x)$.
(8) (M.4) For any $x \neq 0$, there is a $\frac{1}{x}$ with $x \cdot \frac{1}{x} = 1$.
(9) (D) There is a distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$.

Question 4.9. The units 0 and 1 are unique. For any $x \in F$, the inverse $(-x)$ and $\frac{1}{x}$ is unique.

Proof. Exercise. □

Definition 4.10. (The set of rational numbers) Consider the set $\mathbb{Z} \times (\mathbb{Z} \setminus 0)$. Define a relation

$$(p_1, q_1) \equiv (p_2, q_2) \Longleftrightarrow p_1 \cdot q_2 = p_2 \cdot q_1.$$

The relation $\equiv$ is an equivalence relation. Define $\mathbb{Q}$ as the set of the equivalent classes of $\mathbb{Z} \times (\mathbb{Z} \setminus 0)$.

Question 4.11. Let A be a set with an equivalence relation. Show that A is a disjoint union of the set of equivalence classes. Is it true if the relation is not an equivalent relation?

Proof. Exercise. □

Definition 4.12. Let F be a field with order $<$. $(F, <)$ is an order fields if the following conditions holds. Let $x, y, z \in F$.

(1) $y < z \implies x + y < z + x$.
(2) $x > 0, y > 0 \implies xy > 0$.

Theorem 4.13. *Let $(F, <)$ is an ordered field. Let $x, y, z \in F$. We have the following properties,*

(1) *If $x > 0$, then $(-x) < 0$.*
(2) *If $x > 0$, $y < z$, then $xy < xz$.*
(3) *If $x < 0$, $y < z$, then $xy > xz$.*
(4) *If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.*
(5) *If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.*

Proof. (1): Since $x > 0$, $x + (-x) > 0 + (-x)$. Thus $0 < (-x)$. (2): Since $y < z$, $y + (-z) < 0$. Since $x > 0$, $x(y + (-z)) < 0$. Thus $xy < xz$. (3): Similar with the proof of (2). (4): Since $<$ is an order, $x \neq 0$, then $x < 0$ or $x > 0$. If $x > 0$, then $x^2 = x \cdot x > 0$. If $x < 0$, then $-x > 0$, then $x^2 = (-x) \cdot (-x) > 0$. Since $1 = 1^2$, $1 > 0$. (5): Since $y \cdot \frac{1}{y} = 1 > 0$, then $\frac{1}{y} > 0$. The same arguement shows $\frac{1}{x} > 0$. Multiplying $\frac{1}{x} \cdot \frac{1}{y}$ on the both side of $x < y$, we have $\frac{1}{y} < \frac{1}{x}$. □

Proposition 4.14. *The set of rational numbers* $\mathbb{Q}$ *is a field with order*

$$(p_1, q_1) > (p_2, q_2) \Longleftrightarrow p_1 \cdot q_2 - p_2 \cdot q_1 \text{ and } q_1 \cdot q_2 \text{ have the same sign.}$$

Proof.

(1) (Well defined of $<$) Since the set $\mathbb{Q}$ is a set of equivalent classes, we need to check: assume $(p_1', q_1') \equiv (p_1, q_1)$, $(p_2', q_2') \equiv (p_2, q_2)$, and $(p_1, q_1) < (p_2, q_2)$, then $(p_1', q_1') < (p_2', q_2')$. It is clear by definition.
(2) ($<$ is an order) It is because $<$ is an order in $\mathbb{Z}$.
(3) The addition is $+$: $(p_1, q_1) + (p_2, q_2) = (p_1 \cdot q_2 + p_2 \cdot q_1, q_1 \cdot q_2)$. The multiplication $\cdot$ is: $(p_1, q_1) \cdot (p_2, q_2) = (p_1 \cdot p_2, q_1 \cdot q_2)$. The unit of addition is the equivalent class of $(0, 1)$. The unit of multiplication is the equivalent class of $(1, 1)$.

□

Theorem 4.15. *The rational numbers* $(\mathbb{Q}, <)$ *is an ordered fields.*

Proof. Exercise □

5. Lecture 4

In this lecture, we construct the set of real numbers.

Definition 5.1. (sequence) Let A be a set. A sequence of A is a map,

$$f : \mathbb{N} \to A.$$

Equivalently, we denote as $(a_0, a_1, \cdots, a_n, \cdots) = (a_n)$. We write $\lim_{n \to \infty} a_n = 0$ if for any $\epsilon > 0$, there exists N_ϵ, $|a_n| < \epsilon$.

Definition 5.2. (Cauchy sequence of $\mathbb{Q}$) Let $(a_0, a_1, \cdots, a_n, \cdots)$ be a sequence of $\mathbb{Q}$. It is a Cauchy sequence if for any $\epsilon > 0$, there exists an integer N_ϵ such that for any $n, m > N_\epsilon$, $|a_n - a_m| < \epsilon$.

Example 5.3. The sequence $(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots)$ is a Cauchy sequence and $\lim_{n \to 0} a_n = 0$.

Lemma 5.4. *A Cauchy sequence is bounded.*

We write $S_{\mathbb{Q}}$ as the set of Cauchy sequence of $\mathbb{Q}$. Define a relation in $S_{\mathbb{Q}}$ as $(a_n)R(b_n) \Longleftrightarrow \lim_{n \to 0}(a_n - b_n) = 0$.

Theorem 5.5. *The relation* R *in* $S_{\mathbb{Q}}$ *is an equivalent relation.*

Proof.
Reflexive Let $(a_n) \in S_{\mathbb{Q}}$. Since $lim_{n \to \infty}(a_n - a_n) = 0$, $(a_n)R(a_n)$
Symmetric If $(a_n)R(b_n)$, then $(b_n)R(a_n)$. It is by definition.
Transitive If $(a_n)R(b_n)$, $(b_n)R(c_n)$, then for any ϵ, there is a $N_{1,\epsilon}$ such that $|a_n - b_n| < \frac{1}{2}\epsilon$ for $n > N_{1,\epsilon}$. There is a $N_{2,\epsilon}$ such that $|b_n - c_n| < \frac{1}{2}\epsilon$ for $n > N_{2,\epsilon}$. Then, for any $\epsilon > 0$, take $N_\epsilon = Max(N_{1,\epsilon}, N_{2,\epsilon})$.

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Therefore, $(a_n)R(c_n)$. □

Definition 5.6. (The set of real numbers) Define the set of real numbers $\mathbb{R}$ as the equivalent classes of Cauchy sequences of $\mathbb{Q}$.

Usually write a real number as $a.*****$. For example, $3.1426783\cdots$. The corresponding Cauchy sequence is $(3, 3.1, 3.14, 3.142, \cdots,)$. One can show that the digital representation of real numbers is the same with the representation of Cauchy sequences.

Definition 5.7. Let $(a_n), (b_n) \in \mathbb{R}$. Define $(a_n) + (b_n) = (a_n + b_n)$. Define $(a_n) \cdot (b_n) = (a_n \cdot b_n)$.

Proof. Take $\epsilon = 1$. By definition, there exists N, if $n, m > N$, then $|a_n - a_m| < 1$. Let $m = N + 1$, then $|a_n| \leq |a_{N+1}| + 1$. Let $M = Max(a_0, a_1, \cdots, a_N, |a_{N+1})| + 1$. Then, $|a_n| \leq M$ for all $n \geq 0$. □

Theorem 5.8. *The binary operation "+" and "·" is well-defined.*

Proof. Assume $(a'_n) \equiv (a_n)$, then $a'_n + b_n \equiv (a_n) + (b_n)$. This implies the addition "+" is well-defined. We show $(a'_n \cdot b_n) \equiv (a_n \cdot b_n)$, which implies $\cdot$ is well-defined. Since (b_n) is bounded by Lemma5.4, there exists some M such that $|b_n| \leq M$ for all $n \geq 0$. Therefore $|a'_n \cdot b_n - a_n \cdot b_n| = |(a'_n - a_n) \cdot b_n| \leq |(a'_n - a_n) \cdot M|$. Since $\lim_{n \to 0}(a'_n - a_n) = 0$, $\lim_{n \to 0}(a'_n \cdot b_n - a_n \cdot b_n) = 0$. □

Theorem 5.9. *The set of real numbers with addition + and multiplication · is a field.*

Proof. The nontrivial criterion we need to check is the existence of inverse of multiplication.

Lemma 5.10. *Assume $(a_n) \neq 0$. There exists N such that $a_n \neq 0$ for $n > N$. In particular, $a_n > 0$ for all $n > N$ or $a_n < 0$ for all $n > N$.*

Proof. If not, then there is a increasing sequence of positive numbers $(N_k)_{k=1}^{\infty}$ such that $a_{N_k} = 0$. Since (a_n) is a Cauchy sequence, for any ϵ,there exists N such that for $n, m > N$, $|a_n - a_m| < \epsilon$. We may assume $N > N_1 + 1$. Take $m = N_1$, then $|a_n| < \epsilon$ for $n > N$. This implies $(a_n) \equiv 0$, a contradiction. If $a_n > 0$ for all $n > N$ and $a_n < 0$ for all $n > N$ are not not true, then we can also prove $(a_n) = 0$. □

Now, we fix such N with $a_n \neq 0$ for $n > N$. Define a sequence $(c_n) = (a_0, a_1, \cdots, a_N, a_{N+1}^{-1}, a_{N+2}^{-1}, \cdots)$. Then c_n is the multiplicative inverse of a_n. □

Let $(a_n), (b_n) \in \mathbb{R}$. Define $(a_n) < (b_n)$ if there exists some N such that $a_n < b_n$ for $n > N$.

Lemma 5.11. *The relation $<$ of $\mathbb{R}$ is well-defined.*

Proof. If $(a'_n) \equiv (a_n)$, then a'_n is closed to a_n, and $a'_n < b_n$ for large enough n. □

Lemma 5.12. *The relation $<$ is an order in $\mathbb{R}$.*

Proof. Let $(a_n) \neq (b_n) \in \mathbb{R}$. Let $c_n = a_n - b_n$. Then $(c_n) \neq 0$. By Lemma5.10, one and only one of the following happens,

(1) For large enough N, c_n is positive for $n > N$.
(2) For large enough N, c_n is negative for $n > N$.

This implies $(a_n) < (b_n)$ or $(a_n) > (b_n)$. The relation $<$ is clearly transitive. □

Theorem 5.13. $(\mathbb{R}, +, \cdot, <)$ *is an ordered field.*

Proof. Let $(a_n), (b_n), (c_n) \in \mathbb{R}$.

(1) If $(b_n) < (c_n)$, then $(a_n) + (b_n) < (a_n) + (c_n)$ by definition.
(2) If $(a_n) > 0$, $(b_n) > 0$, then $(a_n) \cdot (b_n) > 0$ by Lemma5.10.

□

Theorem 5.14. *Let $E \subset \mathbb{R}$ be a bounded above subset. Then the least upper bound of E in $\mathbb{R}$ exists, which is denoted as $SupE$. Similarly, for the bounded below subset, the greatest lower bound exists.*

Proof. Omitted. □

6. Lecture 5

The set of real numbers is similar with the set of rational numbers, they are both not algebraically closed. For example, consider the equation $x^2 + 1 = 0$.

Lemma 6.1. *There is no $x_0 \in \mathbb{R}$ such that $x_0^2 + 1 = 0$.*

Proof. If $x_0^2 + 1 = 0$, then $x_0^2 = -1 > 0$, contradiction. □

In order to fix the problem, we introduce the formal symbol i for the root of $x^2 + 1 = 0$. Define the set of complex numbers:

Definition 6.2. Define $\mathbb{C} = \mathbb{R}[i] = \{a + bi | a, b \in \mathbb{R}, i^2 = -1\}$

(1) (a+bi)+(c+di)=a+c+(b+d)i
(2) (a+bi)(c+di)=ac-bd+(bc+ad)i

Theorem 6.3. *The set of complex numbers is a field.*

Proof. The inverse of $a + bi$ is $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$. □

Theorem 6.4. *(The fundamental theorem of algebra) The set of complex numbers is algebraically closed. Namely, any polynomial $f(X) \in \mathbb{C}[X]$ has a solution.*

Proof. Omitted. □

Theorem 6.5. *The set of complex numbers is not an order field.*

Proof. Exercise. □

Question 6.6. Is the set $\mathbb{Q}[i]$ a field? Is it algebraically closed?'

Proof. Exercise. □

Let $(a_0, a_1, \cdots, a_n, \cdots)$ be a sequence of real numbers. We write $\lim_{n\to\infty} a_n = a$ if for every $\epsilon > 0$, there is an integer N_ϵ such that $|a_n - a| < \epsilon$. We say the sequence (a_n) converges to a. For example. $\lim_{n\to\infty} \frac{1}{n} = 0$. The constant sequence $(1, 1, \cdots, 1, 1 \cdots)$ converges to 1. If For any positive integer N, there exists n_N such that $|a_n| > N$ for $n > n_N$, then we say the sequence (a_n) diverges. For example, the sequence (n) diverges.

Proposition 6.7. *The convergent sequence converge to a unique real number.*

Question 6.8. Let $a_n = 1$ if n is even; $a_n = 0$ if n is odd. Does (a_n) converges to a real number? Does the sequence (a_n) diverges?

Theorem 6.9. *Assume* $\lim_{n\to\infty} a_n = a$. *Then the set* $\{a_n\}_{n=0,1,2,\cdots}$ *is bounded.*

Theorem 6.10. *Let* (a_n) *and* (b_n) *be two convergent sequence. Assume* $\lim_{n\to\infty} a_n = a$. $\lim_{n\to\infty} b_n = b$. *Then,*

(1) $\lim_{n\to\infty}(a_n + b_n) = a + b$.
(2) $\lim_{n\to\infty} a_n \cdot b_n = a \cdot b$
(3) *Assume* $b \neq 0$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Example 6.11. $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$.

Theorem 6.12. *Let* (a_n) *and* (b_n) *be two sequences. Assume* $\lim_{n\to\infty} a_n = a$, *and the sequence* (b_n) *diverges. Then, the sequecen* $(\frac{a_n}{b_n})$ *converges to* 0.

Proof. According to Theorem 6.9, there is a number M such that $|a_n| \leq M$. For every $\epsilon > 0$, let $N = [\frac{M}{\epsilon}]$. There is an N_ϵ such that $|b_n| > N$ for every $n > N_\epsilon$. This implies $|\frac{a_n}{b_n}| \leq |\frac{M}{b_n}| < \frac{M}{\frac{M}{\epsilon}} = \epsilon$ for $n > N_\epsilon$. □

Example 6.13. $\lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{n^3+1} = 0$.

Definition 6.14. (Cauchy sequence of $\mathbb{R}$) Let $(a_0, a_1, \cdots, a_n, \cdots)$ be a sequence of $\mathbb{R}$. It is a Cauchy sequence if for any real number $\epsilon > 0$, there exists an integer N_ϵ such that for any $n, m > N_\epsilon$, $|a_n - a_m| < \epsilon$.

Theorem 6.15. *Assume* (a_n) *is a Cauchy sequence. There is a real number* a *such that* $\lim_{n\to\infty} a_n = a$.

Theorem 6.16. *If* (a_n) *is a convergent sequence, then* (a_n) *is a Cauchy sequence.*

Proof. For any $\epsilon > 0$, there is a $N_{\frac{\epsilon}{2}}$ such that $|a_n - a| < \frac{\epsilon}{2}$ for $n > N_{\frac{\epsilon}{2}}$. Then,

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \epsilon$$

for $n, m > N_{\frac{\epsilon}{2}}$.

□

7. Lecture 6

Let (a_n) be a sequence in $\mathbb{R}$. A subsequence of (a_n) is a sequence (a_{n_k}) with $n_0 < n_1 < \cdots < n_k < \cdots$.

Example 7.1. Let $a_n = n$. Consider $b_n = a_{2n} = 2n$. It is a subsequence of (a_n).

Theorem 7.2. *Let* (a_n) *be a sequence in* $\mathbb{R}$. *Then* (a_n) *is a convergent sequence if and only if each subsequence of* (a_n) *is a convergent sequence.*

Proof. It suffices to prove each subsequence of (a_n) converges if (a_n) converges. Let $\lim_{n\to\infty} a_n = a$. Then for every $\epsilon > 0$, there is k such that $|a_n - a| < \epsilon$ for $n > k$. For the subsequence (a_{n_k}), $n_k \geq k$, then $|a_{n_j} - a| < \epsilon$ for $j > k$. □

Theorem 7.3. *Let* $E \subset \mathbb{R}$ *be a upper bounded set. Then there is a sequence* (a_n) *that converges to the least upper bound of* E.

Proof. Let a be the least upper bound of E. Then consider $a - \frac{1}{n}$. It is not an upper bound of E since a is the least upper bound. Therefore, there is an a_n such that $a - \frac{1}{n} < a_n \leq a$. This implies $|a_n - a| < \frac{1}{n}$. Consider the sequence (a_n), then $\lim_{n\to\infty} a_a = a$. □

Corollary 7.4. *Let (a_n) be a bounded sequence. There is a subsequence that converges.*

Theorem 7.5. *(Monotone convergent theorem) Let (a_n) be a sequence such that $a_n < a_m$ if $n < m$. Assume it is a bounded sequence, then the sequence (a_n) is a convergent sequence.*

Proof. Exercise. ☐

Theorem 7.6. *Let $(a_n), (b_n), (c_n)$ be sequences in $\mathbb{R}$. Assume $c_n \leq a_n \leq b_n$. Suppose the sequence c_n and (b_n) converges to p, then the sequence a_n converges to p.*

Proof. Take a difference $0 < a_n - c_n < b_n - c_n$, we may assume $c_n = 0$, $\lim_{n\to\infty} b_n = 0$. For $\epsilon > 0$, there is N_ϵ such that $|a_n| < |b_n| < \epsilon$ for $n > N_\epsilon$. ☐

Theorem 7.7. *We have the following limits:*

(1) *If $p > 1$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.*
(2) *If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.*
(3) *If $|p| < 1$, then $\lim_{n\to\infty} p^n = 0$.*
(4) *$\lim_{n\to\infty} \sqrt[n]{n} = 1$.*

Proof. (1): Since $p > 1$, $0 < \frac{1}{n^p} < \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, according to theorem 7.6, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(2): If $0 < p < 1$, consider $\frac{1}{\sqrt[n]{p}}$. Therefore, we assume $p \geq 1$. If $p = 1$, it is trivial. If $p > 1$, denote $a_n = \sqrt[n]{p} - 1$. Then $1 + n \cdot a_n < (1 + a_n)^n = p$. Hence $0 < a_n < \frac{p-1}{n}$. By theorem 7.6, $\lim_{n\to\infty} a_n = 0$, and then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(3) Assume $p > 0$, otherwise, $p^n \leq |p|^n$. We write $p = \frac{1}{1+\eta}$, where $\eta > 0$. Fix a large integer k, we have inequality for $n > 2k$,

$$(1+\eta)^n > \binom{n}{k}\eta^k = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k)}{1 \cdot 2 \cdot 3 \cdots k} \cdot \eta^k > \frac{n^k}{2^k k!} \cdot \eta^k.$$

Then,

$$0 < \frac{1}{(1+\eta)^n} < \frac{2^k k!}{\eta^k} \cdot n^{-k}.$$

According to theorem 7.6, $\lim_{n\to\infty} \frac{1}{(1+\eta)^n} = 0$.

(4): Define $a_n = \sqrt[n]{n}$. Then $n = (1 + a_n)^n > \frac{n(n-1)}{2} \cdot a_n^2$. Therefore,

$$0 < a_n < \sqrt{\frac{2}{n-1}}.$$

According to theorem 7.6, $\lim_{n\to\infty} a_n = 0$. Thus $\lim_{n\to\infty} \sqrt[n]{n} = 1$. ☐

8. Lecture 7

In this lecture, we study the infinite series of real numbers.

Let (a_n) be a sequence of real numbers. Define $s_n = \sum_{i=0}^{n} a_i$. We have a new sequence of real numbers (s_n). Denote $\sum_{i=0}^{\infty} a_i$ as the sequence (s_n). If (s_n) is a convergent sequence that converges to s, we write $s = \sum_{i=0}^{\infty} a_i$. Clearly, $a_n = s_n - s_{n-1}$.

Theorem 8.1. *Let (a_n) be a sequence of real numbers. The series $\sum_{i=0}^{\infty} a_i$ is a convergent sequence if and only if for any $\epsilon > 0$, there is N_ϵ such that*

$$|\sum_{i=n}^{m} a_i| < \epsilon$$

for $n, m > N_\epsilon$.

Proof. The sequence $\sum_{i=0}^{\infty} a_i$ converges if and only if (s_n) is a Cauchy sequence. □

Theorem 8.2. *If the sequence (s_n) converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. Let $n = m$ in theorem 8.1. Then $|a_n| < \epsilon$. □

Question 8.3. If the sequence (a_n) converges to 0, is it true that the sequence (s_n) converges?

Proof. It is not true. For example, let $a_n = \frac{1}{n}$. We have $\lim_{n\to\infty} a_n = 0$, but the infinite series $\sum_{i=0}^{\infty} a_n$ diverges. □

Theorem 8.4. *(Bounded above convergent theorem) Let (a_n) and (c_n) be sequences of real numbers, and assume $|a_n| \le c_n$. Suppose $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.*

Proof. Since $|\sum_{i=n}^{m} a_i| \le \sum_{i=n}^{m} |a_i| \le \sum_{i=n}^{m} c_i$, by Cauchy criterion theorem 6.15 and theorem 6.16, the series $\sum_{n=0}^{\infty} a_n$ converges. □

Corollary 8.5. *(absolute convergent theorem) The infinite series $\sum_{n=0}^{\infty} a_n$ converges if the infinite series $\sum_{n=0}^{\infty} |a_n|$ converges.*

Proof. Let $c_n = |a_n|$ in theorem 8.4. □

Example 8.6. The infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges.

Theorem 8.7. *(A useful criterion for convergence) Let (a_n) be a sequence of real numbers. Suppose $a_0 > a_1 > a_2 > \cdots \ge 0$, then the series $\sum_{n=0}^{\infty} a_n$ converges if and only if the sequence $\sum_{k=0}^{\infty} 2^k \cdot a_{2^k}$ converges.*

Proof. By theorem 7.5, it suffice to consider the boundedness of s_n. We write $b_k = \sum_{n=1}^{k} 2^k \cdot a_{2^k}$. Suppose (b_k) is bounded. Consider $n < 2^k$. Then

$$s_n \le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + a_{2^k+1} + \cdots a_{2^{k+1}-1}) \le a_1 + 2a_2 + 4a_4 + \cdots 2^k \cdot a_{2^k} = b_k.$$

This implies (s_n) is bounded. Suppose (s_n) is bounded. Consider $n > 2^k$. Then

$$s_n \ge (a_1 + a_2) + (a_3 + a_4) + \cdots + (a_{2^{k-1}} + a_{2^{k-1}+1} + \cdots + a_{2^k}) \ge \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k} = \frac{1}{2}b_k.$$

This implies (b_k) is bounded. Therefore, the boundedness of (s_n) and (b_k) are equivalent. Thus, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$ converges. □

Theorem 8.8. *Let $\alpha \in \mathbb{R}$. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$.*

(1) *If $\alpha = 1$, the infinite series diverges.*
(2) *If $\alpha > 1$, the infinite series converges.*

Proof. The sequence $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges(diverges) if and only if $\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^{k\alpha}} = \sum_{k=1}^{\infty} (2^{1-\alpha})^k$ converges(diverges). Thus if $\alpha > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges. If $\alpha < 1$, $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges. □

Theorem 8.9. *The infinite series $\sum_{n=2}^{\infty} \frac{1}{n\cdot(ln(n))^2}$ converges.*

Proof. According to theorem 8.7, it converges if and only if $\sum_{k=2}^{\infty} 2^k \cdot \frac{1}{2^k\cdot(ln(2^k))^2} = \sum_{k=2}^{\infty} \frac{1}{(ln(2))^2\cdot k^2}$ converges. The latter follows from theorem 8.8. □

Theorem 8.10. *The infinite series* $\sum_{n=2}^{\infty} \frac{1}{n \cdot ln(n)}$ *diverges.*

Proof. Exercise. □

Theorem 8.11. *(product) Let* (a_n) *and* b_n *be two sequences of real numbers. Suppose*

(1) *the partial sum* $s_n = \sum_{n=0}^{\infty} a_n$ *is a bounded sequence.*
(2) b_n *is decreasing.*
(3) $\lim_{n\to\infty} b_n = 0$

Then the infinite series $\sum_{n=}^{\infty} a_n \cdot b_n$ *converges.*

Proof. Omitted. □

Corollary 8.12. *(alternate convergent theorem) Let* b_n *be a decreasing sequence that* $\lim_{n\to\infty} b_n = 0$. *Then the infinite series* $\sum_{n=0}^{\infty}(-1)^n b_n$ *converges.*

Proof. Let $a_n = (-1)^n$ in theorem 8.11. □

Example 8.13. The infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Let $(a_n = \frac{1}{n!})$. Since $s_n < 1 + \frac{1}{2^0} + \frac{1}{2^1} + \cdot \frac{1}{2^{n-1}} < 3$, the infinite series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. Define the real number $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Theorem 8.14. $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$.

Proof. Omitted. □

9. Lecture 8

Let $\mathbb{R}$ be the set of real numbers.

Definition 9.1. We have some natural subsets. Let $a, b \in \mathbb{R} \cup \infty$.

(1) (open interval) $\{x|a < x < b\}$
(2) (closed interval) $\{x|a \leq x \leq b\}$.
(3) (open closed interval) $\{x|a < x \leq b\}$;$\{x|a \leq x < b\}$.

Let $A \subset \mathbb{R}$. The A is usually intervals above. Let $f : A \to \mathbb{R}$ be a map. A is the defined domain of the function f. For example, the function $ln(x)$ is defined at $\{x \in \mathbb{R}|x > 0\}$. We write $\lim_{x\to x_0, x>x_0} f(x) = f^+(x_0)$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f^+(x_0)| < \epsilon$$

for $0 < x - x_0 < \delta$. The $f^+(x_0)$ is called the right limit of $f(x)$ at the point x_0. Similarly, we write $\lim_{x\to x_0, x<x_0} f(x) = f^-(x_0)$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f^-(x_0)| < \epsilon$$

for $0 > x - x_0 > -\delta$. The $f^-(x_0)$ is called the left limit of $f(x)$ at the point x_0.

Definition 9.2. Let $f : A \to \mathbb{R}$ be a function. If the left and right limit exists at the point $x_0 \in A$, and if $f^-(x_0) = f^+(x_0)$, then we say $f(x)$ is continuous at the point x_0. If $f(x)$ is continuous at any point $x \in A$, then we say $f(x)$ is a continuous function at A.

Lemma 9.3. *The function $f : A \to \mathbb{R}$ is continuous at the point $x_0 \in A$ if for any $\epsilon > 0$, there is a $\delta > 0$ and $|f(x) - f(x_0)| < \epsilon$ for $|x - x_0| < \delta$.*

Example 9.4. Let $A = \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_n x^x + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $f(x)$ is continuous at any point $x \in \mathbb{R}$.

Example 9.5. Let $f(x) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0. \end{cases}$ The $f(x)$ is continuous at $x \neq 0$, but not at $x = 0$. It is clear $\lim_{x\to 0} f^{-1}(x) = 1$, and $\lim_{x\to 0} f^{+}(x) = 0$.

Theorem 9.6. *Let $f(x)$ and $g(x)$ be continuous functions at x_o. The following functions are continuous,*

(1) $f(x) + g(x)$.
(2) $f(x) \cdot g(x)$.
(3) *Assume* $g(x_0) \neq 0$. $\frac{f(x)}{g(x)}$
(4) $f(g(x))$.

Example 9.7. The elementary functions are continuous,

(1) $f(x) = e^x$
(2) $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.
(3) $f(x) = log_a(x)$.
(4) $f(x) = cos(x)$;$f(x) = sin(x)$.

Example 9.8. The function $f(x) = sin(x) - cos(e^{2x^2 + log_2(x)})$ is continuous at $x \in (0, +\infty)$.

Proposition 9.9. *Define a function $f : \mathbb{R} \to \mathbb{R}$ as follows,*

$f(x) = \begin{cases} 1, & x \text{ is not rational} \\ 0, & x \text{ is rational.} \end{cases}$ *The $f(x)$ is not continuous at any point $x \in \mathbb{R}$.*

Theorem 9.10. *Let $f(x) : A \to \mathbb{R}$ be a function that continuous at $x_0 \in A$. If $f(x_0) \neq y$, then there is an open interval I such that $f(x) \neq y$ for $x \in A \cap I$.*

Proof. Without loss of generality, we assume $y > f(x_0)$. Let $\epsilon = \frac{|y - f(x_0)|}{2}$. Since $f(x)$ is continuous at x_0, there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon$$

for $x_0 - \delta < x < x_0 + \delta, x \in A$. That is,

$$f(x_0) - \frac{y - f(x_0)}{2} < f(x) < f(x_0) + \frac{y - f(x_0)}{2}.$$

Since $y > f(x_0)$, $y > f(x_0) + \frac{y - f(x_0)}{2}$. Then $f(x) < y$ for $x_0 - \delta < x < x_0 + \delta, x \in A$. □

Theorem 9.11. *Let $f(x)$ be a continuous function at the closed interval $[a, b]$. Let c be a real number between $f(a)$ and $f(b)$. Then there is a x_0 such that $f(x_0) = c$.*

Proof. Define a set $S \subset [a, b]$ as $S = x \in [a, b] | f(x) < c$. First, $a \in S$. Next, S is bounded. Let s be the least upper bound of S. We have a sequence (a_n) such that $\lim_n a_n = s$. Therefore $f(s) \leq c$ since f is continuous. If $f(s) \neq c$, then $f(s) < c$. Then $s \in S$. Since f is continuous at s, there is an $s_0 > s \in [a, b]$ such that $f(s_0) < c$. This implies $s_0 \in S$. Since s is the least upper bound of S, $s_0 \leq s$ contradicts that $s_0 > s$. □

Corollary 9.12. *Let $f(x)$ be a continuous function at inverval $[a, b]$. If $f(a) < 0$, $f(b) > 0$, then there is $s \in [a, b]$ such that $f(s) = 0$.*

Theorem 9.13. *Let $f(x)$ be a polynomial of odd degree over $\mathbb{R}$. The $f(x)$ has at least one real root.*

Proof. we write $f(x) = a_n x^{2n+1} + a_{n-1}x^{2n} + \cdots + a_1 x + a_0$. Since $\lim_{x \to +\infty} \frac{x^m}{x^{m+1}} = 0$, $x^{m+1} > x^m$ for large x. Similarly. Therefore, $f(x) > 0$ for large $x \geq b$. Similarly, $f(x) < 0$ for $x \leq a$, a is small enough. Thus, according to theorem 9.12, there is a $s \in \mathbb{R}$ such that $f(s) = 0$. □

Theorem 9.14. *Let $f(x)$ be a continuous function at a, b, then there are $s_{min}, s_{max} \in [a, b]$ such that $f_{s_{min}}$ is the minimal value, $f(s_{max})$ is the maximal value.*

Proof. We prove s_{max} exists. For the existence of s_{min}, we consider the function $-f(x)$. If $f(x)$ is not bounded above, then there is a sequence $(a_n) \subset [a, b]$ such that $f(a_n) > n$. Since $\{a_n\}$ is a bounded set, there is a subsequence (a_{n_k}) that converges to a. Since $f(x)$ is continuous, $\lim_{k \to +\infty} f(a_{n_k}) = f(a)$. However, for any large k, $f(a_{n_k}) > n_k \geq k$, which implies the sequence $(f(a_{n_k}))$ diverges, a contradiction. □

Proposition 9.15. *Theorem 9.14 implies Theorem 9.11.*

10. Lecture 9

Let $f : A \to \mathbb{R}$ be a function. If the limit $\lim_{x \to x_0, x > x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists, then we denote the limit as $f^{'+}(x_0)$. It is the right derivative of f at the point x_0. Similarly, if the limit $\lim_{x \to x_0, x < x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists, then we denote the limit as $f^{'-}(x_0)$. It is the left derivative of f at the point x_0.

Definition 10.1. The f is differentiable at the point x_0 if the left and right derivative exists, and $f^{'+}(x_0) = f^{'-}(x_0)$. If f is differentiable at x_0, w2e call $f'(x_0) = f^{'+}(x_0) = f^{'-}(x_0)$ as the derivative of f at x_0. If f is differentiable at any point $x_0 \in A$, then we say f is differentiable function at A.

Example 10.2. Let $f(x) = x$ be the identity function, then $f(x)$ is differentiable at $x \in \mathbb{R}$.

Proof. The left and the right limit exists, and they are equal to $\lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$. □

Lemma 10.3. *The function f is differentiable at x_0 if and only if there exists $f^{'}(x_0)$, and for any $\epsilon > 0$, there is a $\delta > 0$ such that*

$$|\frac{f(x) - f(x_0)}{x - x_0} - f^{'}(x_0)| < \epsilon$$

for $|x - x_0| < \delta$.

Example 10.4. Let $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$. Then $f^{'+}(0) = 1$, $f^{'-}(0) = -1$. Thus, $f(x)$ is not differentiable at the point 0.

Theorem 10.5. *Let $f(x) = x^n$. Then $f'(x_0) = n \cdot x_0^{n-1}$ for $x_0 \in \mathbb{R}$.*

Proof. Since $\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x} = n \cdot x_0^{n-1} + \Delta x \cdot g(x, \Delta x)$, we have

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = n \cdot x_0^{n-1}.$$

□

Theorem 10.6. *Let $f(x)$ and $g(x)$ be differentiable functions at x_0. The following functions are differentiable at x_0,*

(1) $c \cdot f(x)$
(2) $f(x) + g(x)$.
(3) $f(x) \cdot g(x)$.
(4) *Assume* $g(x_0) \neq 0$. $\frac{f(x)}{g(x)}$
(5) $f(g(x))$.

In particular,

(1) $(c \cdot f(x))^{'} = c \cdot f^{'}(x)$
(2) $(f(x) + g(x))^{'} = f^{'}(x) + g^{'}(x)$
(3) $(f(x) \cdot g(x))^{'} = f^{'}(x) \cdot g(x) + f(x) \cdot g^{'}(x)$.
(4) $(\frac{f(x)}{g(x)})^{'} = \frac{f^{'}(x) \cdot g(x) - f(x) \cdot g^{'}(x)}{(g(x))^2}$
(5) $(f(g(x)))^{'} = f^{'}(g(x)) \cdot g^{'}(x)$.

Theorem 10.7. *The elementary functions are differentiable,*

(1) $f(x) = e^x$
(2) $f(x) = \frac{P(x)}{Q(x)}$, *where $P(x)$ and $Q(x)$ are polynomials.*
(3) $f(x) = log_a(x)$.
(4) $f(x) = cos(x);f(x) = sin(x)$.

In particular,

(1) $(a^x)^{'} = a^x \cdot ln(a)$. $a > 0$.
(2) $(log_a x)^{'} = \frac{1}{x \cdot ln(a)}$. $a > 0$.
(3) $sin(x)^{'} = cos(x)$; $(cos(x))^{'} = -sin(x)$.

Example 10.8. Let $f(x) = e^{5x} \cdot cos(x)$. Then $f^{'}(x) = (e^{5x})^{'} \cdot cos(x) + e^{5x} \cdot (cos(x))^{'} = 5e^{5x} \cdot cos(x) - e^{5x} \cdot sin(x)$

Theorem 10.9. *(Fermat' theorem) Let c be a local minimum of maximum of $f(x)$, and $f(x)$ is differentiable at c. Then $f^{'}(c) = 0$.*

Proof. If not, we may assume $f^{'}(c) > 0$. Then $f^{'+}(c) = f^{'-}(c) > 0$. There are x closed to c that $f(x) > c$ or $f(x) < c$, contradicts that c is the local minimum or maximum. □

Theorem 10.10. *(Rolle theorem) Let f be a continuous function at $[a, b]$, and it is differentiable at (a, b). If $f(a) = f(b)$, then there is a c such that $f^{'}(c) = 0$.*

Proof. According to theorem 9.14, $f(x)$ obtain maximum and minimum at some points in $[a, b]$. If f is a constant function, then the statement is true. we shall assume f is not a constant function. Let $c \in (a, b)$ be maximum point. Then $f^{'}(c) = 0$ by theorem 10.9. □

Theorem 10.11. *(Lagrange mean value theorem) Let $f(x)$ be a continuous function at $[a, b]$ and it is differentiable at (a, b). Then there is a c such that*

$$f(b) - f(a) = f^{'}(c) \cdot (b - a).$$

Theorem 10.12. *Define* $g(x) = f(x) - \frac{f(a)-f(b)}{a-b} \cdot x$*. Then* $g(x)$ *is continuous at* $[a, b]$ *and differentiable at* (a, b)*. Since* $g(a) = g(b)$*, by Rolle theorem(theorem 10.10), there is a* $c \in (a, b)$ *such that* $g^{'}(c) = 0$*. Since* $g^{'}(x) = f^{'}(x) - \frac{f(a)-f(b)}{a-b}$*,* $f^{'}(c) = \frac{f(a)-f(b)}{a-b}$*.*

Theorem 10.13. *If* $f^{'}(x) > 0$ *for* $x \in [a, b]$*, then* $f(x) > f(y)$ *if* $x > y$*. Namely,* $f(x)$ *is increasing. Similarly, if* $f^{'}(x) < 0$*, then* $f(x)$ *is decreasing.*

Proof. By theorem 10.11, there is a c with $f(x)-f(y) = f^{'}(c)(x-y)$, where $c \in [y, x]$. Thus, $f(x) > f(y)$ if $x > y$. □

11. Lecture 10

Theorem 11.1. *(generalized mean value theorem) Let* $f(x)$ *and* $g(x)$ *be a continuous function at* $[a, b]$*, and are differentiable at* (a, b)*, then there is a* $c \in (a, b)$*,*

$$(f(b) - f(a)) \cdot g^{'}(c) = f^{'}(c) \cdot (g(b) - g(a)).$$

Proof. Define $h(x) = (f(b) - f(a)) \cdot g(x) - f(x) \cdot (g(b) - g(a))$. Then $h(a) = h(b)$. According to Theorem 10.10, there is a c such that $h^{'}(c) = 0$. □

Theorem 11.2. *(L'Hospital's rule) Suppose* $f(x)$ *and* $g(x)$ *are differentiable functions at* (a, b)*, where* $-\infty \le a < b \le +\infty$*, and* $g^{'}(x) \neq 0$*. Suppose*

$$\frac{f^{'}(x)}{g^{'}(x)} \to A$$

when $x \to a$*. If*

(1) $f(x) \to 0$ *and* $g(x) \to 0$ *as* $x \to a$*.*
(2) *or* $g(x) \to \infty$*,*

then $\frac{f(x)}{g(x)} \to A$ *as* $x \to a$*. The analogous statement holds for* $x \to b$*.*

Proof. Let $a < x < y < b$. Then $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f^{'}(c)}{g^{'}(c)}$. Let $x \to 0$ and $y \to a$, then $c \to a$. We have $\frac{f(x)}{g(y)} \to A$. □

Example 11.3. $\lim_{x\to 0} \frac{sin(x)}{x} = 1$.

Proof. Since $\lim_{x\to 0} \frac{(sin(x)^{'})}{(x)^{'}} = \lim_{x\to 0} cos(x) = 1$, by L'Hospital's rule, $\lim_{x\to 0} \frac{sin(x)}{x} = 1$. □

Example 11.4. Let $n \ge 1$. Then $\lim_{x\to +\infty} \frac{x^n}{e^x} = 0$.

Proof. Apply L'Hospital's rule n-th times. □

We write $f^{(n)}(x)$ as the n times derivative of $f(x)$ if it exists.

Theorem 11.5. *Let* n *be a positive integer. Suppose* $f^{(n-1)}(x)$ *is a continuous function at* $[a, b]$*, and* $f^{(n-1)}(x)$ *is differentiable at* (a, b)*. Let* α *and* β *be distinct point in* (a, b)*. Assume* $\alpha < \beta$*. Define a polynomial,*

$$P(t) = \sum_{i=0}^{n-1} \frac{f^{(i)}(\alpha)}{i!}(t - \alpha)^i.$$

Then there is a $\alpha < x < \beta$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!} \cdot (\beta - \alpha)^n.$$

Note that $n = 1$, this is just the mean value theorem.

Proof. Define M with

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n.$$

Define $g(t) = f(t) - P(t) - M(t - \alpha)^n$. Then $g(\alpha) = g(\beta) = 0$. Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$, we have,

$$g(\alpha) = g^{(1)}(\alpha) = g^{(2)}(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0.$$

Apply theorem 10.10 to $g(x)$, there is a x_1 such that $g^{'}(x_1) = 0$. Apply theorem 10.10 again to $g^{'}(x)$ with $g^{'}(\alpha) = g^{'}(x_1) = 0$. Then there is a x_2 such that $g^{(2)}(x_2) = 0$. Continuing to $g^{(n-1)}(x)$, there is a $\alpha \leq x \leq \beta$ with $g^{(n)}(x) = 0$. Thus there is x between α and β such that $f^{(n)}(x) = M$. □

If $\lim_{n\to+\infty} \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n = 0$, then $f(x) = \sum_{i=0}^{+\infty} \frac{f^{(i)}(\alpha)}{i!} \cdot (x - \alpha)^i$.

Example 11.6. Let $f(x) = e^x$. Let $\alpha = 0$. For $0 < x < \beta$, we have $0 < |\frac{f^{(n)}(x)}{n!}| \cdot |\beta|^n < \frac{e^\beta \cdot |\beta|^n}{n!}$. Fix the β, then $\lim_{n\to+\infty} \frac{f^{(n)}(x)}{n!} \cdot \beta^n = 0$. Thus we have,

$$e^\beta = \sum_{i=0}^{+\infty} \frac{\beta^i}{i!}.$$

Example 11.7. Since $sin(x)^{(2n)} = (-)^n cos(x)$;$sin(x)^{(2n+1)} = (-1)^n cos(x)$ we have,

$$sin(x) = \sum_{i=0}^{+\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$$

Since $cos(x)^{(2n+1)} = (-1)^n sin(x); cos(x)^{2n} = (-)^n cos(x)$, we have,

$$cos(x) = \sum_{i=0}^{+\infty} \frac{(-1)^i x^{2i}}{(2i)!}.$$

Theorem 11.8. *We have the Euler formula,*

$$e^{ix} = cos(x) + i \cdot sin(x).$$

Proof. $e^{ix} = \sum_{n=0}^{+\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{+\infty} \frac{(i)^n (x)^n}{n!}$. Since $i^{2n+1} = (-1)^n i$, $i^{2n} = (-1)^n$, $e^{ix} = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} i = cos(x) + i \cdot sin(x)$. □

12. Lecture 11

Definition 12.1. Let $[a, b]$ be a closed interval. A partition P of $[a, b]$ is a finite set of points $x_0, x_1, \cdots, x_n$ such that

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b.$$

we write $\Delta x_i = x_i - x_{i-1}$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Consider a partition P. Define,

$$M_i = sup\ f(x), x_{i-1} \le x \le x_i.$$

$$m_i = inf\ f(x), x_{i-1} \le x \le x_i.$$

$$U(P, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i.$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

We define the upper and lower Riemann integrals of f,

$$\overline{\int_a^b} f dx = inf\ U(P, f).$$

$$\underline{\int_a^b} f dx = sup\ L(P, f).$$

If $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$, then we write $f \in \mathcal{R}$, where $\mathcal{R}$ is the set of Riemann integrable functions. We denote the integral as $\int_a^b f dx$.

Remark 12.2. Let α be a monotonically increasing function on $[a, b]$. Write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. We define the Riemann-Stieltjes integrable similarly. If $\alpha(x) = x$, it is the Riemann integral.

Let P' and P be two partitions of $[a, b]$, we say P' is a refinement of P if every points of P is in P'. For example, $a = x_0 \le x_1 \le x_2 \le x_3 = b$ is a refinement of $a = x_0 \le x_2 \le x_3 = b$.

Theorem 12.3. *If P' is a refinement of P, then*

$$L(P, f, \alpha) \le L(P', f, \alpha).$$

$$U(P', f, \alpha) \le U(P, f, \alpha).$$

Proof. It suffice to prove the statements by adding more one point to the partition. Let $x \in (a, b)$. We write P_x for this partition. Let $M_{a,x}$ be the maximal value of f on $[a, x]$. Let $m_{a,x}$ be the minimal value of f on $[x, b]$. Let $M_{x,b}$ be the maximal value of f on $[x, b]$. Let $m_{[a,b]}$ be the minimal value of f on $[x, b]$. Then,

$$U(P_x, f, \alpha) = M_{a,x} \cdot (\alpha(x) - \alpha(a)) + M_{x,b} \cdot (\alpha(b) - \alpha(x)) \le M_x \cdot (\alpha(x) - \alpha(a) + \alpha(b) - \alpha(x)) = M_x \cdot (\alpha(b) - \alpha(a)).$$

Similarly,

$$L(P_x, f, \alpha) \ge m_x \cdot (\alpha(a) - \alpha(b)).$$

□

Theorem 12.4. $\overline{\int_a^b} f dx \ge \underline{\int_a^b} f dx.$

Proof. For two partitions P_1 and P_2, let P' be the common refinement of P_1 and P_2. According to Theorem 12.3,

$$L(P_1, f, \alpha) \le L(P', f, \alpha) \le U(P', f, \alpha) \le U(P_2, f, \alpha).$$

Taking sup over P_1, and inf over P_2, we have the inequality. □

Theorem 12.5. *$f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for any $\epsilon > 0$ there is a partition P such that,*

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

Proof. For any $\epsilon > 0$, there is a P that $U(P,f,\alpha) - \overline{\int}_a^b < \frac{\epsilon}{3}$, and $L(P,f,\alpha) - \underline{\int}_a^b < \frac{\epsilon}{3}$. Since $f \in \mathcal{R}(\alpha)$ if and only if $\overline{\int}_a^b f dx - \underline{\int}_a^b f dx < \frac{\epsilon}{3}$, then $f \in \mathcal{R}(\alpha)$ if and only if there is a P that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. □

Definition 12.6. Let f be a function defined on $[a,b]$. f is uniformly continuous on $[a,b]$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in [a,b], |x-y| < \delta$.

Example 12.7. Let $f(x) = 7x+5$. Then $f(x)$ is uniformly continuous at $\mathbb{R}$.

Proof. For any $\epsilon > 0$, $|f(x) - f(y)| < \epsilon$ for $|x-y| < \frac{\epsilon}{7}$. □

Example 12.8. Let $f(x) = \frac{1}{x}$ define on $(0,1)$. Then $f(x)$ is continuous on $(0,1)$ but not uniformly continuous on $(0,1)$.

Proof. Exercise. □

Proposition 12.9. *Let f be a continuous function on $[a,b]$. Then f is uniformly continuous on $[a,b]$.*

Proof. Omitted. □

Theorem 12.10. *Let f be a continuous function on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$.*

Proof. Let $\epsilon > 0$ be small enough number. There is a δ such that $|f(x)-f(y)| < \frac{\epsilon}{\alpha(b)-\alpha(a)}$ for $|x-y| < \delta$. Let N be a large enough number that $\cdot \frac{\alpha(b)-\alpha(a)}{N} < \delta$. Consider the partition P_n of $[a,b]$ that is uniformly divided by N. Then

$$U(P_N,f,\alpha) - L(P_N,f,\alpha) < N \cdot \frac{\epsilon}{\alpha(b)-\alpha(a)} \cdot \frac{\alpha(b)-\alpha(a)}{N} = \epsilon.$$

According to Theorem 12.5, $f \in \mathcal{R}(\alpha)$. □

Theorem 12.11. *Let $f, g \in \mathcal{R}(\alpha)$ on $[a,b]$.*

(1) *$f+g \in \mathcal{R}(\alpha)$, and $\int_a^b (f+g) d\alpha = \int_a^b f dx + \int_a^b g d\alpha$.*
(2) *$cf \in \mathcal{R}(\alpha)$ and $\int_a^b cf d\alpha = c\int_a^b f d\alpha$.*
(3) *Let $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and $[c,b]$. In particular, $\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$.*
(4) *If $f(x) \le g(x)$ on $[a,b]$, then*

$$\int_a^b f d\alpha \le \int_a^b g d\alpha$$

(5) *If $|f| \le M$, then*

$$|\int_a^b f d\alpha| \le M(\alpha(b) - \alpha(a)).$$

(6) *If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$. In particular,*

$$\int_a^b f d(\alpha_1+\alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2.$$

If c is a positive constant, then $\int_a^b f dc\alpha = \int_a^b cf d\alpha$

(7) *Let* $m \le f(x) \le M$.*Let* $\phi(x)$ *be a continuous function on* $[m, M]$*. Then* $\phi(f(x)) \in \mathcal{R}(\alpha)$ *on* $[a, b]$*.*

Proof. We prove (7). □

Corollary 12.12. *Let* $f, g \in \mathcal{R}(\alpha)$*. Then*

(1) $fg \in \mathcal{R}(\alpha)$.
(2) $|f| \in \mathcal{R}(\alpha)$*. In particular,*

$$|\int_a^b f d\alpha| \le \int_a^b |f| d\alpha.$$

Proof. Let $\phi(x) = x^2$. According to Theorem 12.11, $f^2, g^2, (f+g)^2 \in \mathcal{R}(\alpha)$. Then $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$ is in $\mathcal{R}(\alpha)$. Take $\phi(x) = |x|$, then $|f| \in \mathcal{R}(\alpha)$. □

13. Lecture 12

Let α be an increasing function on $[a, b]$. Assume α is differentiable and $\alpha' \in \mathcal{R}[a, b]$. Let f be a bounded function on $[a, b]$.

Theorem 13.1. $f \in \mathcal{R}(\alpha)$ *if and only if* $f\alpha' \in \mathcal{R}$*. In the case* $f \in \mathcal{R}(\alpha)$*, we have,*

$$\int_a^b f\alpha' dx = \int_a^b f d\alpha.$$

Proof. Let $\epsilon > 0$ be a small enough number. Since $\alpha' \in \mathcal{R}$, there is a partition P with

$$|U(P, \alpha') - L(P, \alpha')| < \epsilon.$$

By mean value theorem, there is a $t_i \in [x_{i-1}, x_i]$ that

$$\Delta\alpha_i = \alpha'(t_i)\Delta x_i.$$

Let $s_i \in [x_{i-1}, x_i]$. Then,

$$\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \epsilon.$$

Let $M = sup(f(x))$. Since $\sum_{i=1}^n f(s_i)\Delta\alpha_i = \sum_{i=1}^n \alpha'(t_i)\Delta x_i$, we have,

$$|\sum_{i=1}^n f(s_i)\Delta\alpha_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)| < M\epsilon.$$

Hence,

$$\sum_{i=1}^n f(s_i)\Delta\alpha_i < U(P, f\alpha') + M\epsilon.$$

holds for any $s_i \in [x_{i-1}, x_i]$. Therefore,

$$U(P, f, \alpha) < U(P, f\alpha') + M\epsilon.$$

Similarly,

$$U(P, f\alpha') < U(P, f, \alpha) + M\epsilon.$$

Then, we have

$$|\overline{\int_a^b} f d\alpha - \overline{\int_a^b} f\alpha' dx| < M\epsilon.$$

Thus, $\overline{\int_a^b} f\alpha' dx = \overline{\int_a^b} f d\alpha$. The same argument holds for $\underline{\int_a^b}$. □

Theorem 13.2. *(Change of variable) Let $\phi(y)$ be a strictly increasing and differentiable function maps $[a,b]$ to $[A,B]$. Let $f \in \mathcal{R}(\phi)$ on $[A,B]$. Then,*

$$\int_A^B f(x)dx = \int_a^b f(\phi(y))\phi'(y)dy.$$

Next, we show the integration is a certain inverse operation of differentiation.

Theorem 13.3. *Let $f \in \mathcal{R}$ on $[a,b]$. Define*

$$F(x) = \int_a^x f(t)dt.$$

Then $F(x)$ is continuous. If f is continuous at some point $x_0 \in [a,b]$, and

$$F'(x_0) = f(x_0).$$

Proof. Since $f \in \mathcal{R}$, f is a bounded function. Let $|f(t)| \leq M$ on $[a,b]$. Then, $|F(y)-F(x)| = |\int_x^y f(t)dt| \leq M|y-x|$.

Therefore $F(x)$ is a continuous function. Next,

$$|\frac{F(x_0+\Delta x) - F(x_0)}{\Delta x} - f(x_0)| = |\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}(f(t) - f(x_0))dt|.$$

Since $f(x)$ is continuous at the point x_0, for any $\epsilon > 0$, there is $\delta > 0$ such that $|f(t) - f(x_0)| < \epsilon$ for $x \in (x_0 - \delta, x_0 + \delta)$. Let $|\Delta x| < \delta$. Then

$$|\frac{F(x_0+\Delta x) - F(x_0)}{\Delta x} - f(x_0)| < \epsilon.$$

□

Theorem 13.4. *(Fundamental theorem of calculus) Let $f \in \mathcal{R}$ on $[a,b]$. If there is a differentiable function $F(x)$ on $[a,b]$ that $F'(x) = f(x)$, then,*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let ϵ be small enough number. Then there is a partition P that

$$U(P,f) - L(P,f) < \epsilon.$$

By Lagrange mean value theorem, there is t_i that

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1}).$$

Therefore $F(b) - F(a) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$ Therefore, $|F(b) - F(a) - \int_a^b f(t)dt| < \epsilon$ for any $\epsilon > 0$. Thus, $\int_a^b f(t)dt = F(b) - F(a)$. □

Corollary 13.5. *(Integration by parts) Let F and G be differentiable function with $F' = f$, $G' = g$. Assume $f, g \in \mathcal{R}$. Then,*

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x).$$

Proof. Let $H(x) = F(x)G(x)$. Use fundamental theorem of calculus of $H(x)$ and $H(x)' = f(x)G(x) + F(x)g(x)$. □

Example 13.6. Let $A = \{(x, y) \in \mathbb{R} | x^2 + y^2 \leq r\}$. The area of A is $4\int_0^r \sqrt{r^2 - x^2}$. Let $x = rsin(\theta)$, $0 \leq \theta \leq \pi$. Then,

$$\begin{aligned} 4\int_0^r \sqrt{r^2 - x^2}dx &= \int_0^{\frac{\pi}{2}} rcos(\theta) \cdot rcos(\theta)d\theta \\ &= 4\int_0^{\frac{\pi}{2}} r^2 \cdot \frac{1 + cos(2\theta)}{2} \\ &= 4r^2 \cdot (\frac{\theta}{2} + \frac{1}{4}sin(2\theta))|_0^{\frac{\pi}{2}} \\ &= \pi r^2 \end{aligned}$$

14. Lecture 13

We write $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. A point in $\mathbb{R}^n$ is denoted as $(a_1, a_2, \cdots, a_n)$, $a_i \in \mathbb{R}$. Define $'+'$ and left multiplication $\cdot$ by $\mathbb{R}$ as follows:

Definition 14.1.

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, ((a_i, (b_i))) \to (a_i + b_i).$$

$$\mathbb{R} \times \mathbb{R}^{\ltimes} \to \mathbb{R}^n, a \cdot (b_i) \to (a \cdot b_i).$$

Lemma 14.2. *We have the following properties,*

(1) $'+'$ *is associative and commutative.*
(2) *For* $a \in \mathbb{R}$, $(b_i), (c_i) \in \mathbb{R}^n$, $a \cdot ((b_i) + (c_i)) = a \cdot (b_i) + a \cdot (c_i)$.
(3) *For* $a, b \in \mathbb{R}, (c_i) \in \mathbb{R}^n$, $(a + b) \cdot (c_i) = a \cdot (c_i) + b \cdot (c_i)$.
(4) *For* $a, b \in \mathbb{R}, (c_i) \in \mathbb{R}^n$, $a \cdot (b \cdot (c_i)) = (a.b) \cdot (c_i)$.
(5) *For* $(b_i) \in \mathbb{R}^n$, $1 \cdot (b_i) = (b_i)$.

The space $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Definition 14.3. Let $\mathbb{F}$ be a field. A vector space V is a set with operation $+$ and multiplication $\cdot$ by $\mathbb{F}$ satisfying the following properties,

(1) $'+'$ is associative and commutative.
(2) For $a \in \mathbb{F}$, $v_1, v_2 \in V$, $a \cdot (v_1 + v_2) = a \cdot (v_1) + a \cdot (v_2)$.
(3) For $a, b \in \mathbb{F}, v \in V$, $(a + b) \cdot v = a \cdot v + b \cdot v$.
(4) For $a, b \in \mathbb{F}, v \in V$, $a \cdot (b \cdot v) = (a.b) \cdot v$.
(5) For $v \in V$, $1 \cdot v = v$.

Example 14.4. Let $V = \mathbb{Q}[i]$. Then V is a vector space over $\mathbb{Q}$.

A subset $W \subset V$ is a sub-vector space if the multiplication and addition map W to W.

Example 14.5. Let $m < n$. Let $E = \mathbb{R}^m \times 0 \subset \mathbb{R}^n$. Then E is a sub-vector space over $\mathbb{R}$.

Example 14.6. Let V be a vector space over $\mathbb{F}$. Let $(v_i)_i^n$ be a collection of vectors. The collection of vectors span a sub-vector space $W = \{w | w = \sum_{i=1}^j a_{n_i} v_{n_i}\}$.

Definition 14.7. A collection of vectors $(v_i)_{i=1}^n$ is a base of V if

(1) They are linearly independent: for any a_{n_i} with $\sum_{i=1}^j a_{n_i} v_{n_i} = 0$, then $a_{n_i} = 0$.

(2) The collection of vectors span the whole vector space V.

Example 14.8. Let $V = \mathbb{R}^n$. Let e_i be the vector that the $i - th$ coordinate is 1, others are zero. Then $(e_i)_{i=1}^n$ is a basis of V.

In this lecture, we always consider the vector space that can be spanned by finitely many vectors. We say the vector spaces are finite dimensional vector space.

Lemma 14.9. *Let V be a finite dimensional vector space over a field $\mathbb{F}$. Then there is a basis of V. The number of vectors in a basis is independent of the choice of bases. We call the number the dimension of V.*

Proof. If not, there are basis $(y_1, y_2, \cdots, y_{n+1})$ and $(x_1, x_2, \cdots, x_n)$. Use cancelation of variable, we deduce y_{n+1} is a linear combination of $(y_1, y_2, \cdots, y_n)$. □

Definition 14.10. Let V and W be vector spaces over $\mathbb{F}$. A map $A : V \to W$ is liner transformation if

(1) A is additive.
(2) A is linear over $\mathbb{F}$, namely $A(a \cdot v) = a \cdot A(v)$.

Choose a base $(e_i)_{i=1}^n$ of V and a base $(v_i)_{i=1}^m$ of W. Then

$$A(e_1) = \sum_{j=1}^{m} a_{1j} \cdot v_j$$
$$A(e_2) = \sum_{j=2}^{m} a_{2j} \cdot v_j$$
$$\cdots$$
$$A(e_i) = \sum_{j=1}^{m} a_{ij} \cdot v_j$$
$$\cdots$$
$$A(e_n) = \sum_{j=1}^{m} a_{nj} \cdot v_j$$

We represent the linear map A as the matrix (a_{ij}).

Example 14.11. Let $V = \mathbb{Q}[i]$ be the $\mathbb{Q}$ vector space. Then $i : V \to V$ by $a \to a \cdot i$ is linear. The matrix of i with respect to basis $\{1, i\}$ is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Definition 14.12. We have two elementary operations for matrix $A = (a_{ij})$ and $B = (b_{ij})$, where $1 \leq i, j \leq n$.

(1) $A + B = (a_{ij} + b_{ij})$.
(2) $A \cdot B = (c_{ij})$, where $c_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$. Note the operation still works for $m \times n$ matrix A and $n \times k$ matrix B.

Theorem 14.13. *Let V_1, V_2, V_3 be vector spaces over $\mathbb{F}$. Let $A_1 : V_1 \to V_2$ and $A_2 : V_2 \to V_3$ be linear maps. Choose bases for V_1, V_2, and V_3, represent the linear maps by corresponding matrix. Then the composition $A_2 \circ A_1$ is represented by the multiplication of the matrix $A_1 \cdot A_2$.*

Proof. We write $(e_{s,i})$ as the bases of V_i, $i = 1, 2, 3$. Then

$$A_1(e_{1,i}) = \sum_{t=1} a^1_{it} e_{2,t} \tag{1}$$

$$A_2(e_{2,i}) = \sum_{t=1} a^2_{it} e_{3,t} \tag{2}$$

Then

$$\begin{aligned} A_2 \circ A_1(e_{1,i}) &= A_2(\sum_{t=1} a^1_{it} e_{2,t}) \\ &= \sum_{t=1} a^1_{it} A_2(e_{2,t}) \\ &= \sum_{t=1} a^1_{it} (\sum_{s=1} a^2_{ts} e_{3,s}) \\ &= \sum_{s=1} (\sum_{t=1} a^1_{it} \cdot a^2_{ts}) e_{3,s} \end{aligned}$$

Then the matrix of the linear map $A_2 \circ A_1$ is $A_1 \cdot A_2$. □

Theorem 14.14. *Let $A : V \to V$ be a linear map. Then A is injective if and only if A is surjective.*

Proof. Let (e_i) be a base of V. If A is a surjection, then $(A(e_i))$ is a base of V. Let $v = \sum_j a_j e_j \in V$ that $A(v) = 0$. Then $\sum_j a_j A(e_j) = 0$. Then $a_j = 0$, and $v = 0$. Thus A is an injection. If A is an injection, then $(A(e_i))$ is a collection of linear independent vectors of V. Since the dimension of V is n, $(A(e_i))$ spans V. □

If A is bijection, we write A^{-1} as the inverse of the map A.

Proposition 14.15. *A^{-1} is a linear map.*

Let E_{ij} be the matrix that commute the i-th row and j-th row of the identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

Lemma 14.16. *The matrix $E_{ij} \cdot A$ is the matrix that commute the i-th row and j-th row of A.*

Let P_{i+j}, j be the matrix that add j-th row to the i-th row of the identity matrix.

Lemma 14.17. *The matrix $p \cdot A$ is the matrix that add the j-th row to the i-th row of A.*

Let Q_i be the matrix that multiplies α at the i-th row of the identity matrix.

Lemma 14.18. *The matrix $Q_i \cdot A$ is the matrix that multiplies α at the i-th row of the matrix A.*

We have similar operations for column, and the corresponding matrix multiplication at the right hand side. We call these matrix the elementary matrix, and the corresponding operation as the elementary operation.

Definition 14.19. Let A be a matrix A viewed as a linear map from $\mathbb{F}^n$ to $\mathbb{F}^n$. We say A is invertible if the map A is a bijection.

Theorem 14.20. *Let A be an invertible matrix. Let $\{E_i\}$ be a sequence of elementary matrix that transform A to I. Namely $E_n E_{n-1} \cdot E_1 A = I$. Then $A^{-1} = E_n E_{n-1} \cdots E_1 I$.*

This gives an algorithm to compute A^{-1}.

Example 14.21. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invetible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

Definition 14.22. Let A be a $n \times m$ matrix. Define the transportation A^T as $m \times n$ matrix, where $a^T_{i,j} = a_{j,i}$.

Theorem 14.23. *We have the following inverse formula:*

(1) $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.
(2) $(A^T)^{-1} = (A^{-1})^T$.

Question 14.24. Let A and B be invertible matrix. Is the matrix $A + B$ invertible?

Theorem 14.25. *Let $A : V \to V$ be a linear map. We write A as the matrix with respect to basis (e_i). Let A' be the matrix with respect to basis (e'_i). Let P be the matrix that $e'_j = p_{ij} e_j$. Then $A' = P \cdot A \cdot P^{-1}$.*

Let $A = (a_{ij})$ be a $n \times n$ matrix. Denote the trace of A as $Tr(A)$, which is defined as $\sum_{i=1}^n a_{ii}$.

Lemma 14.26. *Let A and B be $n \times n$ matrix. Then $Tr(A \cdot B) = Tr(B \cdot A)$.*

Proof. By definition, $Tr(A \cdot B) = \sum_{i,k} a_{ik} \cdot b_{ki}$, $Tr(B \cdot A) = \sum_{k,i} b_{ki} \cdot a_{ik}$. □

Theorem 14.27. *Let $A : V \to V$ be a linear map. We define $Tr(A)$ by choosing a basis. It is independent of the choice of basis.*

Proof. $Tr(P \cdot A \cdot P^{-1}) = Tr(A \cdot P^{-1} \cdot P) = Tr(A)$. □

Example 14.28. Let $i : \mathbb{Q}[i] \to \mathbb{Q}[i]$ be the linear map defined by multiplication. Then $Tr(i) = 0$.

15. Lecture 14

Let $x \in \mathbb{R}^n$. Define a norm of $x = (x_1, x_2, \cdots, x_n)$ as $\sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$.

(1) For $\alpha \in \mathbb{R}$, we have $|\alpha \cdot x| = |\alpha| \cdot |x|$.
(2) $|x + y| \leq |x| + |y|$.

Definition 15.1. Define the open ball $B_\delta = \{x \in \mathbb{R} | |x| < r\}$.

Definition 15.2. Let $f(x_1, x_2, \cdots, x_n) : \mathbb{R}^n \to \mathbb{R}$ be a function. f is continuous at x_0 if for any $\epsilon > 0$, there is a $\delta > 0$ that $|f(x) - f(x_0)| < \epsilon$ for $|x - x_0| < \delta$.

Example 15.3. Let $f(x_1, x_2, \cdots, x_n) = \sum_{i=1} x_i$. Then f is continuous at any point $x_0 \in \mathbb{R}^n$.

let $f(x) : \mathbb{R} \to \mathbb{R}$ be a differentiable function at x_0. Then the following limits exists,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

We write the limit as $f^{'}(x_0)$, then

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - h \cdot f^{'}(x_0)}{h} = 0.$$

Let $r(h) = f(x_0 + h) - f(x_0) - h \cdot f^{'}(x_0)$. Then

$$\lim_{h \to 0} \frac{r(h)}{h} = 0.$$

Equivalently,

$$\lim_{h \to 0} \frac{|r(h)|}{|h|} = 0.$$

Now we let $f(x) : \mathbb{R}^n \to \mathbb{R}$. we still write $||$ for the norm of $\mathbb{R}^n$.

Definition 15.4. Let $f(x) : \mathbb{R}^n \to \mathbb{R}$. The function $f(x)$ is differentiable at $x_0 \in \mathbb{R}^n$ if there is a linear map $f^{'}(x_0) : \mathbb{R}^n \to \mathbb{R}$ that

$$\lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - f^{'}(x_0) \cdot h|}{|h|} = 0.$$

Theorem 15.5. *The linear map $f^{'}(x_0)$ is unique.*

Proof. Suppose another A satisfies the same property with $f^{'}(x_0)$. Then

$$\frac{|(B - f^{'}(x_0)) \cdot h|}{|h|} \le \frac{|f(x_0 + h) - f(x_0) - f^{'}(x_0) \cdot h|}{|h|} + \frac{|f(x_0 + h) - f(x_0) - A \cdot h|}{|h|}$$

If $h \to 0$, then $\frac{|(B - f^{'}(x_0)) \cdot h|}{|h|} \to 0$. Replace h by $t \cdot h$, if $t \to 0$, then $\frac{|(B - f^{'}(x_0)) \cdot t \cdot h|}{|t \cdot h|} \to 0$. There for $A - f^{'}(x_0)$ is a linear map that maps the ball $\{h||h| = 1\}$ to 0. Thus, $A = f^{'}(x_0)$. □

Let $\{e_i\}$ be the standard basis of $\mathbb{R}^n$.

Lemma 15.6. *Let $h = t \cdot e_i$. Let $f(x)$ be a differentiable function at x_0. Then the number $\frac{\partial f(x_0)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_{1,0} + h, x_{2,0}, \cdots, x_{n,0}) - f(x_0)}{h}$ exists. We call the partial derivative of $f(x)$ at x_0*

Theorem 15.7. *Let $f(x) : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function at x_0. Then $f^{'}(x_0) : \mathbb{R}^n \to \mathbb{R}$ is the linear map that maps $h = (h_1, h_2, \cdots, h_n)$ to $(\frac{\partial f(x_0)}{\partial x_1}, \frac{\partial f(x_0)}{\partial x_2}, \cdots, \frac{\partial f(x_0)}{\partial x_n}) \cdot (h_1, h_2, \cdots, h_n) = \sum_{i=1}^n \frac{\partial f(x_0)}{\partial x_i} \cdot h_i$.*

Proof. By the lemma above, $f^{'}(x_0)(e_i) = \frac{\partial f(x_0)}{\partial x_i}$. □

We write $\nabla f(x_0) = \frac{\partial f(x_0)}{\partial x_1} e_1 + \frac{\partial f(x_0)}{\partial x_2} e_2 + \cdots \frac{\partial f(x_0)}{\partial x_n} e_n$. It is the gradient of f. Let u be a unit vector.

Corollary 15.8. *We have directional derivative*

$$\lim_{t \to 0} \frac{f(x_0 + t \cdot u) - f(x_0)}{t} = \nabla f(x_0) \cdot u.$$

In the direction $\nabla f(x_0)$, the value of f decrease or increase the fastest.

Definition 15.9. Let $f(x) : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then f is differentiable if there is a linear map $f^{'}(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ that

$$\lim_{h\to 0} \frac{|f(x_0+h) - f(x_0) - f^{'}(x_0)\cdot h|}{|h|} = 0.$$

Lemma 15.10. *The linear map $f^{'}(x_0)$ is unique.*

Theorem 15.11. *Let $f(x) : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. Then $f^{'}(x_0) = (\frac{\partial f_i}{\partial x_j})$.*

Example 15.12. Let $f(x,y) = (e^y \cdot cos(x), x^2 + y) : \mathbb{R}^2 \to \mathbb{R}^2$. Namely $f_1(x,y) = e^y \cdot cos(x)$, $f_2(x,y) = x^2 + y$. Then $\frac{\partial f_1}{\partial x} = -e^y \cdot sin(x)$, $\frac{\partial f_1}{\partial y} = e^y \cdot cos(x)$. $\frac{\partial f_2}{\partial x} = 2x$, $\frac{\partial f_2}{\partial y} = 1$. Therefore

$$f^{'}(x,y) = \begin{bmatrix} -e^y \cdot sin(x) & -e^y \cdot sin(x) \\ 2x & 1 \end{bmatrix}$$

Theorem 15.13. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to R^N$ be differentiable functions. Let $h = g \circ f$ be the composition. Then,*

$$h^{'}(x) = g^{'}(f(x)) \cdot f^{'}(x).$$

Example 15.14. Let $f : \mathbb{R} \to \mathbb{R}^m$ be a differentiable functioin. It defines a path in R^m. Let $g : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function. Then $h(t) = g(f(t)) = g(x_1(t), x_2(t), \cdots, x_m(t))$.

$$h^{'}(t) = \sum_{i=1}^{m} \frac{\partial g}{\partial x_i} \cdot x_i^{'}(t).$$

Theorem 15.15. *(Taylor expansion) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that $f^{(n)}(X)$ is continuous. Fix a point a. Then*

$$f(a+x) = \sum_{k=1}^{n-1} \frac{1}{k!} \sum D_{i_1 i_2 \cdots i_k} f(a) x_{i_1} x_{i_2} \cdots x_{i_k} + r(x),$$

where $\lim_{x\to 0} \frac{|r(x)|}{|x|^{n-1}} = 0$

Proof. Consider the line $a + tx$. Define $h(t) = f(a+tx)$. Then $h(0) = f(a)$, $h(1) = f(a+x)$. Clearly $h^{(k)}(t) = \sum D_{i_1 i_2 \cdots i_k} f(a+tx) x_{i_1} x_{i_2} \cdots x_{i_k}$. Use Taylor expansion for $h(t)$. □

16. Lecture 15

Let $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be a fontinuous function. We study the Riemann integral of $f(x,y)$ on a Rectangle $\{x, y | a \le x \le b; c \le y \le d\}$. In the case of one variable, we define the upper and lower integrals.

Lemma 16.1. *$f(x,y)$ is a bounded function on the rectangle $\{x, y | a \le x \le b; c \le y \le d\}$.*

We let P_x be a partition of $a \le x \le b$, and let P_y be a partition of $c \le y \le d$. Then define the partition of $\{x, y | a \le x \le b; c \le y \le d\}$ as $P_x \times P_y$. We write P as this partition.

Definition 16.2. Define $U(f,P) = \sum_i \sum_j f_{ij,Max} \Delta A_{i,j}$. Define $L(f,P) = \sum_i \sum_j f_{ij,min} \Delta A_{i,j}$, Here ΔA_{ij} is the volume of the rectangle $\{(x,y) | x_{i-1} \le x \le x_i; y_{j-1} \le y \le y_j\}$.

Definition 16.3. Let S be the set of partitions that are product of partition of x axis and y axis. Define

(1) $\overline{\int\int} f(x,y)dxdy = \liminf_{P\in S} U(f,P)$
(2) $\underline{\int\int} f(x,y)dxdy = \limsup_{P\in S} L(f,P)$

Definition 16.4. Let f be a continuous function. f is Riemann integrable if $\bar{\iint} f(x,y)dxdy = \underline{\iint} f(x,y)dxdy$.

Theorem 16.5. *A continuous function is Riemann integrable on Rectangle.*

Theorem 16.6. *Similar with the case of one variable function, we have,*

(1) *If* $\iint f_1 + f_2 dxdy = \iint f_1 dxdy + \iint f_2 dxdy$.
(2) $\iint c \cdot f dxdy = c \cdot \iint f dxdy$
(3) *If* $f_1 \leq f_2$, *then* $\iint f_1 dxdy \leq \iint f_2 dxdy$.
(4) *If* $|f| \leq M$, *then* $\iint f dxdy \leq M \cdot (b-a) \cdot (d-c)$.
(5) $|\iint f dxdy| \leq \iint |f| dxdy$.

Theorem 16.7. *(Fubini theorem)*

$$\iint f(x,y)dxdy = \int_c^d (\int_a^b f(x,y)dx)dy = \int_a^b (\int_c^d f(x,y)dy)dx.$$

Example 16.8. Let $f(x,y) = xy$ on $[0,1] \times [0,1]$. By Fubini theorem,

$$\iint_{[0,1]\times[0,1]} f(x,y)dxdy = \int_0^1 (\int_0^1 xydx)dy = \int_0^1 \frac{y}{2}dy = \frac{1}{4}.$$

Next, we study the integration along a general region. Let $D = [a \leq x \leq b] \times [g_1(x) \leq y \leq g_2(x)]$. Then we define $\iint_D f(x,y)dxdy$ using partition that cover the region D.

Theorem 16.9.

$$\iint_D f(x,y)dxdy = \int_a^b (\int_{g_1(x)}^{g_2(x)} f(x,y)dy)dx.$$

Similarly, let $D = [f_1(y) \leq x \leq f_2(y)] \times [c,d]$. Then

$$\int_c^d (\int_{f_1(y)}^{f_2(y)} f(x,y)dx)dy.$$

Example 16.10. Let $D = [0,1] \times \{x^3 \leq y \leq \sqrt{x}\}$. Let $f(x,y) = 4xy$. Then,

$$\begin{aligned}\iint_D 2xydxdy &= \int_0^1 (\int_{x^3}^{\sqrt{x}} 2xydy)dx \\ &= \int_0^1 x(x - x^6)dx \\ &= \int_0^1 x^2dx - \int_0^1 x^7dx \\ &= \frac{1}{3} - \frac{1}{8} \\ &= \frac{5}{24}\end{aligned}$$

Example 16.11. Let $D = \{y^3 \leq x \leq \sqrt{y}\} \times [0,1]$. Let $f(x,y) = 2xy$. Then

$$\iint_D 2xydxdy = \frac{5}{24}.$$

Definition 16.12. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\det A = ad - bc$. In the exercise, we know A is invertible if and only if $\det A \neq 0$.

Let $v : \mathbb{R}^2 \to \mathbb{R}^2, (u,v) \mapsto (v_1(u,v), v_2(u,v))$ be a invertible differentiable function. Assume $v^{'}(x)$ is invertible. The basic examples are the invertible linear maos.

Theorem 16.13. *(change of variables) Let D be a region. Let $D^{'} = v(D)$ be another region. Then,*

$$\iint_{D'} f(x,y)dxdy = \iint_{D} f(v_1(u,v), v_2(u,v)) \cdot |\det(v^{'})| dudv.$$

Example 16.14. A useful change of variable function is $x = rcos(\theta)$, $y = rsinc(\theta)$, $0 \leq \theta < 2\pi$, $r \geq 0$. Assume D' is the region of D under change of the above variables. Then,

$$\iint_{D'} f(x,y)dxdy = \iint_{D} f(rcos(\theta), rsin(\theta))rd\theta dr.$$

Example 16.15. Let $f(x,y) = cos(x^2+y^2)$. Let $D = [-1,1] \times \{y| -\sqrt{1-x^2} \leq y \leq 0\}$. Under Polar coordinate (θ, r), we have $\pi \leq \theta \leq 2\pi$, $0 \leq r \leq 1$. Then,

$$\begin{aligned}\iint_{D} cos(x^2+y^2)dxdy &= \int_0^1 \int_{\pi}^{2\pi} rcos(r^2)drd\theta \\ &= \frac{\pi}{2} \cdot sin(1)\end{aligned}$$

We may write $dxdy = rdrd\theta$. It records change of volume form if we change the variables. In general, $dxdy = |Jac(g(u,v))|dudv$, where $Jac(g(u,v))$ is the determinant of the matrix $g^{'}(u,v) : \mathbb{R}^2 \to \mathbb{R}^2$.